where $t = 1/r$. This equation is then solved by iteratively searching for the zero value of the function φ

$$\varphi = -a^t + 1 + bt \tag{13}$$

The form of φ is shown in Fig. 1, and it can be seen that φ has a maximum at the point $t_m = \ln(b/\ln a)/\ln a$ and that $2t_m$ is a reasonable first approximation for t. For any $t_0 > t_m$ the Newton-Raphson scheme

$$t_{j+1} = t_j - \varphi_j'/\varphi_j, \quad \varphi' = -a^t \ln a + b \tag{14}$$

converges to the correct solution, computer experiments showing that about four steps (for $a = 0.98$) are required for full convergence.

The general minimization algorithm is then as follows. Start:

$$x_0, \quad f_0 = f(x_0), \quad g_i = \nabla f_0, \quad p_0 = g_0$$

Iterate: Obtain α_i from: $\partial f(x_i + \alpha_i p_i)/\partial\alpha_i = 0$

$$x_{i+1} = x_i + \alpha_i p_i, \quad f_{i+1} = f(x_{i+1}), \quad g_{i+1} = \nabla f_{i+1}$$

Obtain t_i from: $(f_{i+1}/f_i)^{t_i} = 1 + \alpha_i p_i{}^T g_i t_i / 2f_i$

$$\begin{aligned} \gamma_i &= (f_i/f_{i+1})^{1-t_i}, \quad \beta_i = g^T{}_{i+1} g_{i+1}/g_i{}^T g_i \\ p_{i+1} &= g_{i+1} + \beta_i \gamma_i p_i \end{aligned} \tag{15}$$

In algorithm (15) it was assumed that $f = 0$ at the minimum. If at the minimum $f = c$, f should be replaced by $f - c$ in algorithm (15). If only an estimate of c is used, the algorithm will assume that f is quadratic near (depending on the closeness of $f_{\min}$ to c) the minimum. If $f_{\min} > 0$, but $c = 0$, algorithm (15) will minimize $f = 1/2r[(x - \xi)^T K(x - \xi) + f_{\min}]^r$.

Numerical Example

The present algorithm was used for minimizing

$$f = (\tfrac{1}{2}x^T Kx + x^T b + \tfrac{1}{4})^r$$

$$K = \begin{pmatrix} 4.5 & 7 & 3.5 & 3 \\ 7 & 14 & 9 & 8 \\ 3.5 & 9 & 8.5 & 5 \\ 3 & 8 & 5 & 7 \end{pmatrix} \qquad b = \begin{vmatrix} -0.5 \\ -1.0 \\ -1.5 \\ 0 \end{vmatrix}$$

for $r = \frac{1}{2}$ and $r = 2$. The case $r = 2$ was also tested in Ref. 1. In both cases the starting point was (4, 4, 4, 4), f attaining its minimal value, $f = 0$, at (0.5, −0.5, 0.5, 0). Calculation was carried out in double precision (14 decimal places). With the present algorithm, f was minimized ($f = 10^{-14}$) in no more than five steps, the fifth step resulting from round-off errors. For $r = 2$ the conjugate gradient scheme of Fletcher and Reeves required 69 iterations to reach $f = 10^{-14}$. For $r = \frac{1}{2}$, 300 iterations were needed to reduce f to 10^{-14}. In both cases iteration was not restarted. Generally, restarting the iterations in the Fletcher and Reeves scheme may reduce the total number of steps required for convergence.

References

[1] Jacobson, D. H. and Oksman, W., "An Algorithm that Minimizes Homogeneous Functions of N Variables in N + 2 Iterations and Rapidly Minimizes General Functions," TR 618, Oct. 1970, Harvard Univ., Div. of Engineering and Applied Physics, Cambridge, Mass.

[2] Davidon, W. C., "Variable Metric Method for Minimization," Research and Development Rept. ANL-5990, 1959, Atomic Energy Commission.

[3] Fletcher, R. and Powell, M. J. D., "A Rapidly Convergent Descent Method for Minimization," *The Computer Journal*, Vol. 6, 1963, pp. 163–168.

[4] Fletcher, R. and Reeves, C. M., "Function Minimization by Conjugate Gradients," *The Computer Journal*, Vol. 7, 1964, pp. 149–152.

[5] Schmit, L. A., Bogner, F. K., and Fox, R. L., "Finite Deflection Structural Analysis Using Plate and Shell Discrete Elements," *AIAA Journal*, Vol. 6, No. 5, May 1968, pp. 781–791.

[6] Fox, R. L. and Stanton, E. L., "Development in Structural Analysis by Direct Energy Minimization," *AIAA Journal*, Vol. 6, No. 6, June 1968, pp. 1036–1042.

Approximate Analytic Solution for the Position and Strength of Shock Waves about Cones in Supersonic Flow

SANFORD S. DAVIS*
NASA Ames Research Center, Moffett Field, Calif.

SIMPLE analytic equations are derived for the position and strength of the shock wave emitted by an axisymmetric cone in a steady supersonic flow. These equations are valid for higher Mach numbers and/or larger cone semiangles than the equations derived previously by Lighthill[1] and Whitham.[2] Even though the complete solution of the conical flow problem is well documented in various tables and computer programs, an analytic expression is useful for preliminary design estimates and for ascertaining the relative effects of the various parameters on the shock wave. In addition, these equations may have an important effect on shock wave calculations when applied to sonic boom problems. Recent experiments[3] have shown that theoretical sonic boom signatures emitted by bodies at higher Mach numbers and/or lower fineness ratios are generally longer than the corresponding experimentally determined signatures, that is, the shock wave stands out too far ahead of the freestream Mach cone emitted by the vertex of the body. The results of this Note tend to alleviate this lengthening effect by placing the shock wave closer to the undisturbed Mach cone from the vertex.

The equation derived in this Note is obtained by applying both the Whitham procedure for calculating a uniformly valid first-order solution, and the simplification afforded by the conical symmetry of the flow. In this derivation, only the details of the second order characteristics† are used in order to obtain a solution. Any further increase in accuracy or increases to higher Mach numbers (and/or larger cone semiangles) would necessarily have to include complicated third-order effects on the characteristics.

The Whitham theory is a powerful tool for the determination of the strength and location of weak shock waves in a supersonic flowfield. The theory is based on the premise that a uniformly valid first-order expression for the perturbation quantities can be obtained by fitting the first order perturbations to the second-order characteristics (or a suitable approximation to them). Consider, for example, the disturbance field created by a symmetric cone, of semiangle δ, placed in a steady, supersonic stream flowing with a velocity V. Let the speed of propagation of small disturbances in the free stream be denoted by a_∞, and define the Mach number M and the quantity β by V/a_∞ and $[(V/a_\infty)^2 - 1]^{1/2}$, respectively. A cylindrical coordinate system (x,r,θ) is introduced with its origin at the vertex of the cone, and the x axis along the cone's axis of symmetry. With respect to this

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* NCR Postdoctoral Research Associate; now Research Scientist, Aerodynamics Branch.

† The second-order characteristic represents an expansion of the exact Mach angle up to terms linear in the perturbation velocities.

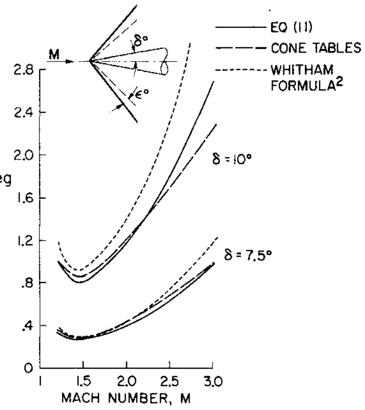

Fig. 1 Variation of shock angle with Mach number for a slender cone.

system, the flow is independent of the polar angle and may be examined in any plane θ = constant.

The perturbation velocities u and v, in the axial and radial directions respectively, are introduced as the first-order deviations of the actual velocities from the steady, freestream flow. The velocity ratios for a cone[2] may be written as

$$\frac{u}{V} = -\delta^2 \int_0^{x-\beta r} \frac{dx_1}{[(x - x_1)^2 - \beta^2 r^2]^{1/2}} \tag{1a}$$

$$\frac{v}{V} = \frac{\delta^2}{r} \int_0^{x-\beta r} \frac{(x - x_1)dx_1}{[(x - x_1)^2 - \beta^2 r^2]^{1/2}} \tag{1b}$$

In order to apply the Whitham theory, these quantities must be expressed in terms of the coordinates $\xi = x - \beta r$ and r

$$\frac{u}{V} = -\delta^2 \int_0^{\xi} \frac{dx_1}{[(\xi - x_1)(\xi - x_1 + 2\beta r)]^{1/2}} \tag{2a}$$

$$\frac{v}{V} = \frac{\delta^2}{r} \int_0^{\xi} \frac{(\xi - x_1 + \beta r)dx_1}{[(\xi - x_1)(\xi - x_1 + 2\beta r)]^{1/2}} \tag{2b}$$

The essence of Whitham's theory lies in the contention that ξ should not be equated to the quantity $x - \beta r$, but instead to the second-order counterpart of this quantity. This second-order correction is easily obtainable from an integration of an expansion for the slope of the exact characteristic line in terms of the perturbation velocities u and v (e.g., see Ref. 2). In this manner an expression for the second-order characteristic, along which the quantity ξ is constant, is obtained as

$$x - \beta r = \xi + \frac{(\gamma + 1)M^4\delta^2}{2\beta^2} \int_0^{\xi} \left(\frac{\xi - x_1 + 2\beta\delta\xi}{\xi - x_1}\right)^{1/2} dx_1 - \frac{(\gamma + 1)M^4\delta^2}{2\beta^2} \int_0^{\xi} \left(\frac{\xi - x_1 + 2\beta r}{\xi - x_1}\right)^{1/2} dx_1 - 2M^2 \int_0^{\xi} \ln \left[\frac{(\xi - x_1 + 2\beta r)^{1/2} - (\xi - x_1)^{1/2}}{(\xi - x_1 + 2\beta\delta\xi)^{1/2} - (\xi - x_1)^{1/2}} \left(\frac{\delta\xi}{r}\right)^{1/2}\right] dx_1 \tag{3}$$

In order to obtain this expression, the integration was extended from the body surface $r' \simeq \delta\xi$ to an arbitrary field point $r' = r$. If the parameter ξ could be eliminated from Eqs. (2) and (3), a uniformly valid first-order approximation to the flowfield can be obtained. In practice, the solution given by Eqs. (2) and (3) is left in parametric form.

Usually, Eq. (3) is simplified by using the asymptotic forms of u and v with respect to the small parameter ξ/r. This procedure is justified by the fact that the quantity ξ/r is considered small both at the front shock wave and far away from the body. It can be shown that the last term, which is customarily neglected in the approximation $\xi/r \to 0$, is indeed negligible as far as the second-order characteristics are concerned. But the second term on the right side of Eq. (3), which represents a near-field effect neglected in the Whitham theory, may be just as important as the first term ξ. (This term arises from the lower limit in the integration.) At first inspection the importance of this term may not be recognized, for the coefficient of ξ is $0(1)$, while the coefficient of the second term is $0(\delta^2)$ (δ is the small parameter). But this second term is multiplied by the factor M^4/β^2, which, at moderate Mach numbers, brings the total coefficient of this second term to first-order importance. In terms of the magnitudes of the relevant quantities, this lower limit term is important when $M^2\delta/\beta = 0(1)$. If the approximation $\xi/r \to 0$ is now applied to Eqs. (2) and (3), the form of the uniformly valid first-order solution for the velocity field is:

$$\frac{u}{V} = -\delta^2 \int_0^{\xi} \frac{dx_1}{(\xi - x_1)^{1/2}(2\beta r)^{1/2}}$$

$$\frac{v}{V} = \frac{\delta^2}{r} \int_0^{\xi} \frac{\beta r dx_1}{(\xi - x_1)^{1/2}(2\beta r)^{1/2}}$$

$$x - \beta r = \xi + \frac{(\gamma + 1)M^4\delta^2}{2\beta^2} \int_0^{\xi} dx_1 - \frac{(\gamma + 1)M^4\delta^2}{2\beta^2} \int_0^{\xi} \frac{(2\beta r)^{1/2}dx_1}{(\xi - x_1)^{1/2}}$$

After integrating, the final result becomes

$$u/V = -\delta^2 (2\xi/\beta r)^{1/2} \tag{4a}$$

$$v/V = \beta\delta^2 (2\xi/\beta r)^{1/2} \tag{4b}$$

$$x - \beta r = \xi + \frac{(\gamma + 1)M^4\delta^2\xi}{2\beta^2} - \frac{(\gamma + 1)(2)^{1/2}M^4\delta^2}{\beta^{3/2}} (r\xi)^{1/2} \tag{4c}$$

The parametric representation shown above is not physically meaningful since the expressions are not single valued functions of position. In order to obtain a single valued solution, a shock wave must be inserted into the flowfield. Following the approximation outlined by Whitham,[2] (i.e., the direction of the shock wave bisects the angle formed by the two Mach lines which contribute to the shock), the shock location is given by

$$x = \beta r - H(r) \tag{5}$$

where $H(r)$ is the solution of the following pair of coupled equations containing the dependent variables $H(r)$ and $\xi(r)$

$$dH(r)/dr = [(\gamma + 1)M^4\delta^2]/(2^{3/2}\beta^{3/2})(\xi/r)^{1/2} \tag{6a}$$

$$H(r) = \frac{(\gamma + 1)2^{1/2}M^4\delta^2}{\beta^{3/2}} (r\xi)^{1/2} - \xi - \frac{(\gamma + 1)M^4\delta^2\xi}{2\beta^2} \tag{6b}$$

The first of Eqs. (6) is obtained from the bisection rule and the second from combining Eq. (5) with the last of Eqs. (4).

This differential equation for $H(r)$ and $\xi(r)$ may be reduced to a single algebraic relation by taking advantage of the conical symmetry of the flow. The shock wave location is determined by the parameter $\eta_{sh} = \xi/r$ which is independent of r. In terms of η_{sh} Eqs. (6) become

$$dH(r)/dr = [(\gamma + 1)M^4\delta^2]/(2^{3/2}\beta^{3/2})(\eta_{sh})^{1/2} \tag{7a}$$

$$H(r) = r\left[\frac{(\gamma + 1)2^{1/2}M^4\delta^2}{\beta^{3/2}} (\eta_{sh})^{1/2} - \eta_{sh} - \frac{(\gamma + 1)M^4\delta^2}{2\beta^2} \eta_{sh}\right] \tag{7b}$$

From the second of Eqs. (7) $dH(r)/dr$ is equal to the bracketed term on the right side (since η_{sh} is independent of r). If this term is equated to the value of $dH(r)/dr$ as given by the first of Eqs. (7) an equation for η_{sh} is obtained as:

$$\eta_{sh}^{1/2}\left\{\frac{3(\gamma+1)M^4\delta^2}{2^{3/2}\beta^{3/2}} - \eta_{sh}^{1/2}\left[1+\frac{(\gamma+1)M^4\delta^2}{2\beta^2}\right]\right\} = 0 \tag{8}$$

Disregarding the trivial solution gives

$$\eta_{sh} = \frac{9(\gamma+1)^2M^8\delta^4}{8\beta^3}\Big/\left[1+\frac{(\gamma+1)M^4\delta^2}{2\beta^2}\right]^2 \tag{9}$$

From Eqs. (9) and (4) the pressure rise across the shock wave is determined as

$$\frac{\Delta p}{p_\infty} = -\gamma M^2\frac{u}{V} = \frac{(\frac{3}{2})[\gamma(\gamma+1)M^6\delta^4/\beta^2]}{1+[(\gamma+1)M^4\delta^2/2\beta^2]} \tag{10}$$

and the angle that the conical shock wave makes with the linearized Mach cone is

$$\epsilon = \frac{(\frac{3}{8})[(\gamma+1)^2M^6\delta^4/\beta^3]}{1+[(\gamma+1)M^4\delta^2/2\beta^2]} \tag{11}$$

Equations (10) and (11) differ from the results of Lighthill and Whitham only by the appearance of the additional $(\gamma + 1)M^4\delta^2/2\beta^2$ factor in the denominators. The validity of these equations easily can be checked by comparing results given by them with those obtained by numerically integrating the exact adiabatic equations of motion.[3] The angular difference (ϵ) between the shock cone and the undisturbed Mach cone is shown in Fig. 1 for two cone semiangles, 7.5° and 10°, as a function of Mach number. Results given by Eq. (11) are compared with those obtained from Whitham's formula[2] and from cone tables.[4] For the conditions illustrated, results from the present study show significantly better agreement with results from the cone tables, especially at the higher Mach numbers. As mentioned previously, any greater accuracy would have to be obtained by resorting to a uniformly valid second-order theory.

References

[1] Lighthill, M. J., "The Position of the Shock Wave in Certain Aerodynamic Problems," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 1, Pt. 3, Sept. 1948, pp. 309–318.

[2] Whitham, G. B., "The Flow Pattern of a Supersonic Projectile," *Communications on Pure and Applied Mathematics*, Vol. V, 1952, pp. 301–348.

[3] Shrout, B. L., Mack, R. J., and Dollyhigh, S. M., "A Wind Tunnel Investigation of Sonic-Boom Pressure Distributions of Bodies of Revolution at Mach Numbers 2.96, 3.83, and 4.63," TN D-6195, 1971, NASA.

[4] Sims, J. L., *Tables for Supersonic Flow Around Right, Circular Cones at Zero Angle of Attack,* NASA SP-3004, 1964.

Stress Concentration in a Cylindrical Shell with an Elliptical Cutout

Ole Tingleff*

Technical University of Denmark, Lyngby, Denmark

Nomenclature

$a^{\alpha\beta}$ = metric tensor
b = half the focal width of the ellipse
B = bending moment at the boundary
c = half the major axis of the ellipse
d = axis ratio for the ellipse
D_i = arbitrary constant
$d_{\alpha\beta}$ = curvature of the shell
E = Young's modulus
$e^{\alpha\beta}$ = alternating tensor
F = auxiliary function
$G(x^1)$ and $H(x^2)$ = factors of the auxiliary function F
h = shell thickness
N = component of boundary force along the boundary normal
n^α = boundary normal
p = internal pressure
P_n, P_t, P_s, and P_b = factors appearing in the components of the boundary forces
R = cylinder radius
S = component of boundary force along the shell normal
s = a hole parameter
T = component of boundary force along the boundary tangent
t^α = boundary tangent
w = deflection along the shell normal
(x,y) = generator coordinate system
(x^1,x^2) = ellipse system
$Z^{\mathrm{I}},Z^{\mathrm{II}}$ = Bessel function, first and second kind
$\mathrm{ceh}_i(x^2)$, $\mathrm{seh}_i(x^2)$ = periodic Mathieu functions
$\alpha_{\mathrm{membr}}, \alpha_{\mathrm{tot}}$ = stress concentration factors
ϕ = complex stress function
ψ = Airy's stress function
$\psi|_\alpha$ = covariant derivative of ψ
$\Delta\psi$ = first invariant of the second derivatives of ψ
ν = Poisson's ratio

Subscripts

$,x,\alpha$ = partial derivative
unif = uniform axial tension
membr = membrane
tot = total
pl = plane

Introduction

Although many papers have been published on the subject of stress concentration at circular and nearly circular holes in cylindrical shells, it appears that the only two that treat elliptical holes are those by Venkitapathy[2] and Murthy.[3] In both cases the major axis of the ellipse is oriented along a generator to the cylinder. In the paper by Venkitapathy the solution of the differential equation is incomplete; compare Eq. (2.17) of Ref. 1 with Eq. (10) of the present Note. Murthy uses a perturbation method, the parameter of which describes the size of the hole, and in the paper he states that the solution applies only to small holes.

The present Note also deals with a circular cylindrical shell weakened by an elliptical hole, but here it is the minor axis that is oriented along a generator to the cylinder. The elastic stress concentration due to axial tension is examined. As the analysis is based on shallow shell theory, the Note begins with an outline of that theory. The general solution of the differential equation is found after a simplifying variable substitution. The boundary conditions, treated with the help of Fourier analysis, together with assumptions regarding the symmetry and asymptotic behavior of the solution, then determine a unique stress field. This method can in certain cases be applied with confidence to holes up to

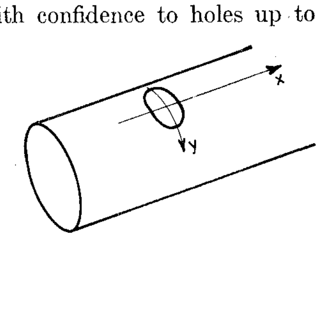

Fig. 1 Cylindrical shell with elliptical cutout and generator coordinate system (x, y).

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Index category: Structural Static Analysis.

* Lecturer at the Institute for Numerical Analysis.